COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA **24** (1) (2004) 35–51

ON A PROBLEM OF CAMERON'S ON INEXHAUSTIBLE GRAPHS

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Received June 15, 2000
Revised July 19, 2001

A graph G is inexhaustible if whenever a vertex of G is deleted the remaining graph is isomorphic to G. We address a question of Cameron [6], who asked which countable graphs are inexhaustible. In particular, we prove that there are continuum many countable inexhaustible graphs with properties in common with the infinite random graph, including adjacency properties and universality. Locally finite inexhaustible graphs and forests are investigated, as is a semigroup structure on the class of inexhaustible graphs. We extend a result of [7] on homogeneous inexhaustible graphs to pseudo-homogeneous inexhaustible graphs.

1. Introduction

In [6], Cameron concludes Section 1.1 with the following problem: which countable graphs G have the property that deleting any vertex of G results in a graph that is isomorphic to G? (This is equivalent to asking which G have the property that deleting any finite subset of vertices of G results in a graph isomorphic to G.) The countable null and complete graphs both trivially satisfy this property. Graphs with the stated property were named *inexhaustible* by Fraïssé [8], and have been studied by Pouzet [13], and El-Zahar and Sauer [7] (in [7] inexhaustible graphs are called *strongly inexhaustible*). Unfortunately, there are $2^{\aleph_0}$-many countable inexhaustible

Mathematics Subject Classification (2000): 05C75, 03C15

The authors gratefully acknowledge support from the Natural Science and Engineering Research Council of Canada (NSERC).

graphs (see Subsection 3.1 below), so Cameron's problem may have no simple resolution. Nevertheless, inexhaustible *homogeneous* graphs (a graph is homogeneous if every isomorphism between finite induced subgraphs is induced by an automorphism) have been classified in [7]. (The result in [7] applies to classes in any relational language.) However, since there are $\aleph_0$-many countable homogeneous graphs (see [12]), the overall classification of the countable inexhaustible graphs is far from complete.

A nontrivial example of a countable inexhaustible graph is the infinite random graph R. In fact, R satisfies the stronger *pigeonhole property*: for each partition of $V(R)$ into sets A and B, the subgraph induced by at least one of A or B is isomorphic to R (see [3] and [5] for more on this property). In this article, in our attempt to answer Cameron's question, we provide a series of negative results which we believe demonstrate that the class of all inexhaustible graphs, written $\mathcal{I}$, is "unclassifiable". We supply $2^{\aleph_0}$-many examples of countable inexhaustible graphs which share many of the properties of R (universality, adjacency properties, existence of one- and two-way hamiltonian paths) but none of which are isomorphic to R (see Theorem 4.1). We prove that $\mathcal{I}$ is not first-order axiomatizable and is not closed under unions of chains (see Theorem 4.2). We prove in Theorems 2.1 and 2.2 that $\mathcal{I}$ supports a semigroup structure that fails to possess many of the familiar properties of semigroups. We consider locally finite countable inexhaustible graphs in Section 3, where a characterization of certain classes of inexhaustible forests seems tractable; see Theorem 3.5.

In Section 5, we present a classification of the countable pseudo-homogeneous graphs in $\mathcal{I}$. This gives a generalization of Theorem 2 of [7] to pseudo-homogeneous graphs, and allows us to prove in Theorem 5.4 that the pseudo-homogeneous G-colourable graphs are inexhaustible.

All our graphs are simple and countable, and will be considered up to isomorphism. If G is a graph and $S \subseteq V(G)$, the *induced subgraph* of G on S is written $G \restriction S$. If G is an induced subgraph of H, we say that H is an *extension* of G or that H *extends* G; we write $G \le H$. For $S \subseteq V(G)$, $G - S$ is the graph formed by deleting every vertex in S and all edges in G incident with vertices in S; if $S = \{x\}$, we write $G - S = G - x$. For graphs $A \le G, H$ so that $V(G) \cap V(H) = V(A)$, the *union of* G, H *over* A, written $G \cup H$, is the graph formed by taking the union of the vertices and edges of G and H. The disjoint union of G and H is written $G \uplus H$. The chordless *circuit* of order n is denoted C_n (C_n is also called the chordless *cycle* of order n). For background on relational structures we direct the reader to Fraïssé [8]; for background in model theory the reader is directed to Hodges [10]. The *age* of a countable graph, written *age*(G), is the set of isomorphism types of finite induced subgraphs of G. The cardinality of the natural numbers is written $\aleph_0$ and $2^{\aleph_0}$ is the cardinality of the real numbers. For cardinals α, β, define $\alpha \cdot G$ to be the disjoint union of α-many copies of G, K_α to be the complete graph of order α, and $K_{\alpha,\beta}$ to be the complete bipartite graph with vertex classes of orders α and β.

2. The substitution monoid

Let G and H be graphs, and let $\{H_x : x \in V(G)\}$ be a set of disjoint copies of H indexed by $V(G)$. We form a new graph, $G \circ H$, the *substitution* of H into G, by deleting each vertex x of G and replacing x with H_x, so that a vertex of H_x is joined to a vertex of H_y if and only if x and y are joined in G. Substitution may be thought of as an operation on the class of countable graphs $\mathcal{G}$; it is easy to see that this operation is associative and has a unit, the trivial graph K_1. Hence, $(\mathcal{G}, \circ)$ is a monoid, which we call the *substitution monoid*, denoted by $\mathcal{G}$. (We note that substitution is often referred to as the *lexicographic product* of G and H.)

The monoid $\mathcal{G}$ has an interesting connection with inexhaustible graphs as shown by the following theorem. Recall that if $A = (A, \cdot)$ is a semigroup then $\varnothing \neq B \subseteq A$ is a *subsemigroup* if B itself is a semigroup; B is a *left ideal* if $A \cdot B \subseteq B$ (a *right ideal* is defined in a similar fashion). We refer the reader to Howie [11] for further background on semigroups.

Theorem 2.1. *Let $\mathcal{I}' \subseteq \mathcal{G}$ be the subset of countable inexhaustible graphs. Then $\mathcal{I}' = (\mathcal{I}', \circ)$ is a left ideal of $\mathcal{G}$ (thus, a subsemigroup).*

Proof. Let $G \in \mathcal{G}$, $H \in \mathcal{I}'$ and fix $x \in V(G \circ H)$. Then x is in some copy H_z of H in $G \circ H$. Let $f : H_z - x \to H_z$ be an isomorphism. The map $F : (G \circ H) - x \to G \circ H$ that is f on $H_z - x$ and the identity otherwise, is an isomorphism. This establishes that $\mathcal{I}'$ is a left ideal. ∎

Even though Theorem 2.1 has a simple proof, notice that it supplies an immediate proof that $\mathcal{I}'$ is closed under disjoint unions and total joins; indeed, replacing the vertices of *any* countable graph by an inexhaustible graph yields an inexhaustible graph.

The following theorem gives more detailed information about the monoid $\mathcal{I}'$. Recall that a semigroup $S = (S, \cdot)$ is *left cancellative* if it satisfies

$$x \cdot y = x \cdot z \ \Rightarrow y = z$$

for all $x, y, z \in S$. *Right cancellative* is defined similarly. The semigroup S is *regular* if every $x \in S$ is *regular*, that is, there is a $y \in S$ so that

$$x \cdot y \cdot x = x.$$

An element $x \in S$ is *idempotent* if $x \cdot x = x$. The semigroup S has a *zero element*, 0, if $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$. A semigroup is *left (right) simple* if it has no proper left (right) ideals.

Theorem 2.2. *1. The class $\mathcal{I}'$ has no zero element.*
2. The class $\mathcal{I}'$ is neither left nor right simple.
3. The class $\mathcal{I}'$ is neither left nor right cancellative.
4. The class $\mathcal{I}'$ is not regular.
5. The class $\mathcal{I}'$ contains $2^{\aleph_0}$-many elements that are not a product of finitely many idempotents.

Proof. (1) Assume that $\mathcal{I}'$ has a zero element H. Then H must have infinitely many connected components as $\overline{K_{\aleph_0}} \circ H = H$. However, since $K_{\aleph_0} \circ H = H$, H is connected of diameter ≤ 2, which is a contradiction.

(2) Let $\mathcal{J}$ be the class of inexhaustible 1-*existentially closed* or 1-*e.c.* graphs: graphs with the property that each vertex is joined to some vertex, and not joined to a vertex other than itself (see the first paragraph of Section 4 below). Note that $\mathcal{J}$ is not empty, as the infinite random graph R is in $\mathcal{J}$, and $\mathcal{J} \neq \mathcal{G}$ since $K_{\aleph_0} \in \mathcal{G} \setminus \mathcal{J}$. Fix $G \in \mathcal{G}$, and $H \in \mathcal{J}$. We claim that $G \circ H$ and $H \circ G$ are 1-e.c. To see this, note that any vertex of $G \circ H$ is in some copy of H, and since H is 1-e.c., x is joined to some vertex, and not joined to a vertex other than itself in that copy. If x is a vertex in $H \circ G$, then suppose that $x \in G_a$ for some $a \in V(H)$. Since H is 1-e.c., then there are $b, c \in V(H)$ so that $ab \in E(H)$ and $ac \notin E(H)$ and $a \neq c$. Then x is joined to every vertex of G_b and to no vertex of G_c.

(3) If we let $G = K_{\aleph_0}$, $H = \overline{K_{\aleph_0}}$, and $J = K_{\aleph_0,\aleph_0}$, then $G \circ H = G \circ J$ is the complete $\aleph_0$-partite graph, with each vertex class infinite (we leave this as an exercise). Let $H = \overline{K_{\aleph_0}}$, $G = K_{\aleph_0}$ and J be the disjoint union of $\aleph_0 \cdot K_2$ and $\overline{K_{\aleph_0}}$. Then J is inexhaustible and $H \circ G = J \circ G = \aleph_0 \cdot K_{\aleph_0}$.

(4) To show that $\mathcal{I}'$ is not regular, we show that R is not a regular element. We prove the stronger property that there is no $H \in \mathcal{I}'$ so that $H \circ R = R$.

Certainly H cannot be $\overline{K_{\aleph_0}}$, as then $H \circ R$ is disconnected and R is connected (of diameter 2). Suppose that H has an edge, say $ab \in E(H)$. Fix $x, y \in V(R_a)$ and $z \in V(R_b)$. If $H \circ R = R$, then $H \circ R$ is n-e.c. for all $n \geq 1$ (see Section 4). In particular, there is a vertex $c \in V(H \circ R) \setminus \{x, y, z\}$ so that c is joined to x but not to y or z. If c is not in $V(R_a)$, then either c is joined to both of x and y or to neither x nor y. Therefore, $c \in V(R_a)$. But then c is joined to z, which is a contradiction.

(5) We will demonstrate in Section 3 below that there are $2^{\aleph_0}$-many countable inexhaustible forests. Let F be a fixed countable inexhaustible

forest that has at least one edge. We show that F is not a product of idempotents, and thereby prove item (5).

Claim. For any graphs $G_1,\ldots,G_n$, we have that $G_i \leq G_1 \circ \cdots \circ G_n$, for each $1 \leq i \leq n$.

The claim follows by induction. If $n=1$, then there is nothing to prove, so consider $n=2$. For each $x \in V(G_1)$, $(G_2)_x$ is a copy of G_2. If we choose a transversal from $(V((G_2)_x) : x \in V(G_1))$ (that is, choose exactly one vertex from each set $V((G_2)_x)$), then the subgraph induced by the transversal is G_1. The case for $n \geq 3$ is similar to the case $n=2$, using the associativity of substitution, and so is omitted.

Now suppose that $F = G_1 \circ \cdots \circ G_n$, for some $G_i \in \mathcal{I}'$ so that $G_i \circ G_i = G_i$ for all $i \in \{1,\ldots,n\}$. Let $j \in \{1,\ldots,n\}$ be the first index so that $G_j \neq \overline{K_{\aleph_0}}$ (there is such an index, otherwise F is null, contrary to the hypothesis). Let $ab \in E(G_j)$ be fixed. Since $G_j \circ G_j = G_j$, the vertices a and b in both $(G_j)_a$ and $(G_j)_b$ are all pairwise joined, so that K_4 is a subgraph of $G_j \circ G_j = G_j$. By the Claim, $G_j \leq F$, so that F contains a copy of K_4. This is a contradiction. ∎

We believe that Theorem 2.2 supports the view expressed in the Introduction that $\mathcal{I}'$ is a badly behaved class: from a semigroup-theoretic view, $\mathcal{I}'$ fails to have many of the basic properties a semigroup can have. An interesting problem that we cannot solve is to classify the idempotents of $\mathcal{I}'$.

3. The locally finite case

Throughout this section, G will always be a countable locally finite graph (that is, each vertex of G has finite degree).

Definition 3.1. 1. The *age-closure* of G is the disjoint union of $\aleph_0$-many copies of each finite induced subgraph of G.

2. The graph G is *age-closed* if there are induced subgraphs G' and H of G, with G' isomorphic to the age-closure of G, such that G is the disjoint union of G' and H.

In other words, G is age-closed if G contains an induced subgraph G' isomorphic to the age-closure of G with the property that there is no edge between a vertex of G' and a vertex not in G'.

We consider the following example, which will both illustrate the notion of an age-closed graph, and help motivate the next theorem. Let G be the infinite one-way path. The graph G is not age-closed, since K_1 is an induced subgraph of G, but each copy H of K_1 in G is joined to vertices not in H.

The age-closure of G, written G', consists of the disjoint union of infinitely many copies of each finite path. Note that G is not inexhaustible, but the disjoint union of G and G' is age-closed and inexhaustible.

Recall that if $x \in V(G)$ then the *neighbour set* of x, written $N(x)$, is the set of all vertices joined to x. The elements of $N(x)$ are called the *neighbours* of x.

Theorem 3.2. *If G is inexhaustible then G is age-closed. The converse holds if G has no infinite components.*

Proof. If S is a finite induced subgraph of G, then consider

$$T = \Big(\bigcup_{x \in V(S)} N(x) \Big) \setminus S.$$

Then T is finite, so $G - T \cong G$. Hence, G contains a "disjoint" copy of S: for all $x \in V(G) \setminus V(S)$, x is not incident with a vertex of S. If there were only finitely many disjoint copies of S, then we could delete these and obtain a graph containing no disjoint copy of S, which is a contradiction.

For the converse, we enumerate the finite distinct induced subgraphs of G as $(S_i : i \geq 1)$, in such a way that the orders of the S_i are monotone increasing (this is possible as there are only finitely many graphs of any given order). In particular, $S_1 = K_1$. Since G is age-closed, G is isomorphic to $\aleph_0 \cdot (\biguplus_{i \geq 1} S_i)$ ($\biguplus$ denotes disjoint union). Fix $x \in V(G)$. Without loss of generality, we can assume that $x \in V(S_j)$ for some $j \geq 1$. If $j = 1$, then deleting x leaves infinitely many copies of K_1 so $G - x \cong G$. If $j > 1$, then deleting x from S_j results in some S_i with $i < j$. Since there are infinitely many S_i and S_j, $G - x \cong G$. ∎

Recall that $\Delta(G)$ is the supremum of the set of degrees of vertices of G. For an integer $i \geq 0$, we define $[i]_G = \{x \in V(G) : \deg(x) = i\}$ (if G is clear from context we will simply write $[i]$).

Theorem 3.3. *Let G be an inexhaustible graph. Then for each integer $i \geq 0$ that is less than or equal $\Delta(G)$, the set $[i]_G$ is infinite.*

Proof. The proof proceeds by cases.

Case 1. Suppose that $\Delta(G)$ is an integer n.

If $[n]$ is finite, then $G - [n]$ has no vertices of degree n, which contradicts inexhaustibility. If $n = 0$, then $G = \overline{K_{\aleph_0}}$, hence, we may assume that $n > 0$.

Now fix i an integer so that $0 \leq i < n$. Suppose that $[i]$ is finite with m elements. If we define $S = \bigcup_{x \in [i]} N(x)$, then S is clearly finite. Suppose that for all $z \in [n]$, z is joined to some vertex in S. Then the set of edges incident with vertices of S is infinite, which is a contradiction. Hence, there is some vertex z in $[n]$ not joined to any vertex of S.

Let $N(z)=\{y_1,\ldots,y_n\}$. Consider $G-\{y_1,\ldots,y_{n-i}\}=H$. Then the degree of z in H is i. But since no y_i is in S, each vertex in $[i]_G$ has degree i in H. Hence, in H, the set $[i]_H$ has at least $m+1$ elements, which contradicts the fact that $G\cong H$.

Case 2. $\Delta(G)=\aleph_0$.
Claim 1. For all $i\geq 0$, the set $[i]$ is nonempty.

We prove the claim by induction on i. If $i=0$, fix $x\in V(G)$. Then x has degree 0 in $G-N(x)\cong G$.

Assume that the Claim is true for i. For $i+1$, consider a vertex z in G of degree $\geq i+1$. Delete sufficiently many neighbours of z, so z then has degree $i+1$ in the remaining graph, which is isomorphic to G. The claim follows.

Now, for a fixed i, define $S=(\bigcup_{x\in[i]}N(x))\setminus[i]$.

Claim 2. There is some $j>i$ and a $y\in[j]$ so that y is not joined to any vertex of S.

Otherwise, by Claim 1 there are infinitely many vertices not in $[i]$ joined to a vertex of S. But S is finite, so it has only finitely many neighbours.

If $[i]$ is finite, then choose y as in Claim 2. The same contradiction may be derived as was found in Case 1. ∎

As we mentioned above, the disjoint union of an infinite one-way path (or a *ray*) and its age-closure is inexhaustible. However, the disjoint union of an infinite two-way path (or a *double ray*) and its age-closure is not inexhaustible. To help explain these facts, we recall the following cardinal-invariant of a locally finite graph:

$$\epsilon(G)=\sup\{\#\text{infinite components of } G-S : S\subset V(G); |S|<\aleph_0\}.$$

The cardinal $\epsilon(G)$ is the *number of ends* of G (see [9]). The proof of the following lemma is left as an exercise.

Lemma 3.4. *Let G be an inexhaustible graph with only finitely many infinite components $G_1,\ldots,G_n$. Then $\epsilon(G_i)=1$, for all $1\leq i\leq n$ (each G_i is one-ended).* ∎

The following theorem characterizes a certain class of locally finite inexhaustible forests. A vertex of degree 1 is called an *end-vertex*.

Theorem 3.5. *Let F be a locally finite forest with only finitely many infinite components T_i, for $1 \leq i \leq n$. Suppose that each T_i has only finitely many end-vertices or none. Then F is inexhaustible if and only if each T_i is a ray and F is age-closed.*

Proof. The proof of the reverse direction is straightforward. Let F be inexhaustible, and let $T = T_i$ be a component of F.

Case 1. The set [1] is empty in T.

In this case, fix $x \in V(T)$. It is easily seen that we can find two infinite disjoint rays emanating from x, which would contradict Lemma 3.4.

Case 2. The set [1] has a unique element 0.

It is straightforward to see that 0 is the terminal vertex of some ray $\mathcal{R}$. We label the vertices of $\mathcal{R}$ by the natural numbers. If some vertex $m > 0$ has degree > 2 then we can find an infinite ray $\mathcal{R}'$ disjoint from $\mathcal{R}$, which would contradict Lemma 3.4. Hence, T must be a ray. We argue this way for all components T_i and obtain the conclusion of the forward direction.

Case 3. The set $[1] = \{x_1, \ldots, x_r\}$ with $r > 1$.

By König's lemma there is a ray $\mathcal{R}$ with end-vertex x_1. Each x_i with $2 \leq i \leq r$ is connected to $\mathcal{R}$ by a unique path $P(i)$ incident with $\mathcal{R}$ at a unique vertex we name z_i. Since $\epsilon(T) = 1$ there is no ray disjoint from $\mathcal{R}$ that is incident with any $P(i)$. Hence, each $P(i)$ and its branches not on $\mathcal{R}$ form a finite tree $T(i)$. See Figure 1.

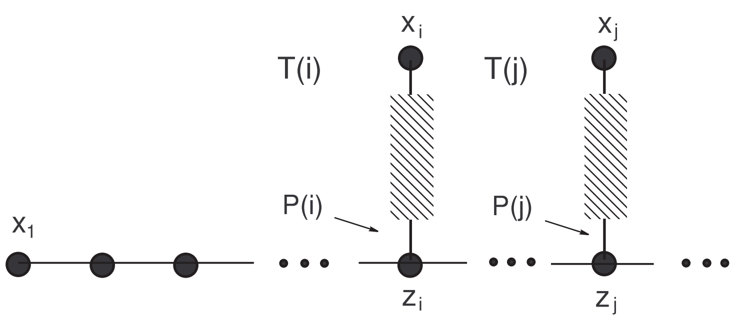

Figure 1. $\mathcal{R}$ and its branches.

If we delete the vertices in $T(i)\backslash\{z_i\}$ for $2 \le i \le r$, then the remaining graph is a ray. We argue in this manner for each T_i, so each infinite component of F is a ray. ∎

We cannot answer the following problem: if we drop the condition on the number of end-vertices in Theorem 3.5, then does the conclusion still hold?

3.1. $2^{\aleph_0}$ many locally finite inexhaustible forests with infinitely many infinite components

The following example demonstrates that an extension of Theorem 3.5 to allow for $\aleph_0$-many infinite components may not be tractable. Let X be a set of infinite subsets of the natural numbers ≥ 1 with pairwise finite intersection, so that $|X| = 2^{\aleph_0}$. For each $S \in X$, we define a forest $F(S)$ as follows. Enumerate S as $\{s_i : i \ge 1\}$ with the s_i strictly increasing. Define $F(S)''$ to consist of $\aleph_0$-many disjoint copies of the following graph $F(S)'$: consider a ray $\mathcal{R}$ indexed by the natural numbers, so that each $i \ge 1$ is incident with exactly s_i disjoint double rays $\mathcal{R}_i$. See Figure 2.

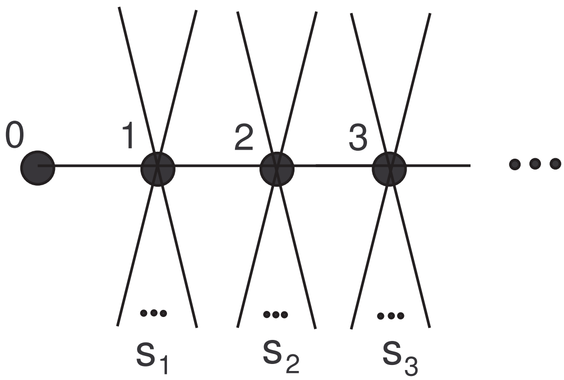

Figure 2. $F(S)'$.

Define $F(S)$ by first adding to $F(S)''$ $\aleph_0$-many disjoint copies of $F(S)' - Y$ for each finite subset Y of $V(F(S)')$, and then taking the age-closure. It is clear that $F(S)$ is locally finite and is inexhaustible. If S and T are distinct elements of X, then we prove that $F(S)$ and $F(T)$ are non-isomorphic.

Let $\{n_i : i \ge 1\} = S\backslash T$. Then $F(S)'$ is never a component of $F(T)$. To see this, note first that $F(S)'$ is not isomorphic to $F(T)'$, since there is no vertex x of degree $2n_1+2$ in $F(T)'$: otherwise, since each $n_i \geq 1$, then x would have to be one of the integers $i \geq 1$ on $\mathcal{R}$; but then x would be incident with n_1 double rays $\mathcal{R}_i$, a contradiction. (In particular, there are no vertices of degree $2n_i+2$ in $F(T)'$.) The only other possibility is if $F(S)'$ is isomorphic to $F(T)'-Y$ for some finite Y. Deleting Y can only decrease the degrees of at most finitely many of the vertices of $F(T)'$. Hence, if we choose j large enough, there will be no vertex of degree $2n_j+2$ in $F(T)'-Y$.

4. Continuum many inexhaustible approximations of R

For a fixed integer $n \geq 1$, a graph G is called *n-existentially closed* or *n-e.c.* if for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex in T and to no vertex in $S \setminus T$. The reader will recall that the infinite random graph R is the unique countable graph that is n-e.c. for all $n \geq 1$ (see [5]). A countable graph which satisfies only finitely many (or equivalently, a single) n-e.c. conditions is said to be an *approximation* of R. We recall that R is *universal*: it embeds all countable graphs as induced subgraphs. The graph R is also *indivisible*: each partition of $V(R)$ into two classes forces one of the classes to contain the original graph as an induced subgraph. Note that any extension of R is indivisible and universal. (See [5] for more details on these and other properties of R.)

The present section is primarily devoted to the proof of the following theorem.

Theorem 4.1. *Let $n \geq 2$ be a fixed integer. Then there are $2^{\aleph_0}$-many non-isomorphic approximations of R having the following properties:*

1. *inexhaustibility;*
2. *n-e.c. but not $(n+1)$-e.c.;*
3. *universality;*
4. *indivisibility;*
5. *has one- and two-way hamiltonian paths.*

Proof. Define $\mathcal{C} = \{C_i : i \geq n+2\}$, and define $\mathcal{X} = \{X : X \subseteq \mathcal{C}$ and $|X| = |\mathcal{C} \backslash X| = \aleph_0\}$. Then $|\mathcal{X}| = 2^{\aleph_0}$, and all the elements of $\mathcal{C}$ are pairwise non-embeddable. Fix $X \in \mathcal{X}$ and enumerate X as $(C_{i_j} : j \geq 1)$.

Define $G(n,X)^0$ to be the graph $\biguplus_{j \geq 1} C_{i_j} \uplus K_{n+1}$ along with the age-closure of this graph. As our inductive hypothesis, assume that $G(n,X)^m$ has been defined so that it is countable and contains $G(n,X)^0$ as an induced subgraph.

List all the $\leq (m+1)$-vertex induced subgraphs of $G(n,X)^m$ as $\{A_i : i \geq 1\}$. Fix $i \geq 1$. List all extensions of A_i to an $(|A_i|+1)$-vertex graph as $B_{1,i}, \ldots, B_{k_i,i}$, *except* those extensions of A_i which would extend one of the C_{i_j} in $G(n,X)^0$ to a wheel (the new vertex would be joined to all the vertices of C_{i_j}) or would extend one of the copies of K_{n+1} to K_{n+2}.

Without loss of generality, we may assume that for each $1 \leq j \leq k$, $V(B_{j,i}) \cap V(G(n,x)^m) = V(A_i)$. Form the union of $G(n,X)^m$ and the $B_{i,j}$ over A_i to form $G(n,X)^{m+1}(i)$. Form the unions of the graphs $G(n,X)^{m+1}(i)$ over $G(n,X)^m$ to form $G(n,X)^{m+1}$. The graph $G(n,X)^{m+1}$ satisfies the inductive hypotheses.

Define

$$G(n,X) = \bigcup_{m \geq 1} G(n,X)^m.$$

By construction, $G(n,X) - V(G(n,X)^0)$ is m-e.c. for all $m \geq 1$; hence, $G \restriction (V(G(n,X)) \setminus V(G(n,X)^0) \cong R$. Since $R \leq G(n,X)$, $G(n,X)$ is universal and indivisible. By construction, $G(n,X)$ is n-e.c.: at each stage of the construction of $G(n,X)$, all n-vertex subsets are extended in the following stage of construction in all possible ways.

However, $G(n,X)$ is not $(n+1)$-e.c., because there is no vertex in $G(n,X)$ joined to each vertex in any copy K of K_{n+1} in $G(n,X)^0$. (The same argument holds for all circuits C_{i_j} in $G(n,X)^0$.) To see this, note that first, no such vertex exists in $G(n,X)^0$. Assume that there is no such vertex in $G(n,X)^m$. By construction, each vertex of $G(n,X)^{m+1}$ is joined to at most a proper subset of $V(K)$. (Note that the vertices added in $G(n,X)^m(i)$ are joined only to the vertices of some A_i in $G(n,X)^{m-1}$ and to no others.)

We show that $G(n,X)$ has a one-way hamiltonian path; the existence of a two-way path is proved similarly. Enumerate $V(G(n,X)) = \{x_i : i \geq 1\}$. Set P_1 to be the induced subgraph by $\{x_1\}$. Suppose that P_n has been constructed so that it is a path containing $x_1, \ldots, x_n$ (and possibly other vertices), with end-vertices x_1 and some other vertex z_n. If $x_{n+1} \in V(P_n)$, then set $P_{n+1} = P_n$ with $z_{n+1} = z_n$.

Consider the case when $x_{n+1} \notin V(P_n)$. If x_{n+1} is joined to z_n, then set $P_{n+1} = G(n,X) \restriction (V(P_n) \cup \{z_{n+1}\})$, and let $z_{n+1} = x_{n+1}$. So consider when x_{n+1} and z_n are not joined. There is some m so that $(V(P_n) \cup \{x_{n+1}\}) \subseteq G(n,X)^m$. Then in $G(n,X)^{m+1}$ we can find a vertex y joined to z_n and x_{n+1}, and to no other vertices in $V(P_n) \cup \{x_{n+1}\}$ (since this is not one of the forbidden extensions). Set P_{n+1} to be the path P_n along with the edges zy, yx_{n+1}, with $z_{n+1} = x_{n+1}$. The union of the chain of the P_n subgraphs is a spanning path of $G(n,X)$ with initial vertex x_1.

The remaining desired properties are proven for the graphs $G(n,X)$ through a series of claims.

Claim 1. If $X,Y\in\mathcal{X}$ and $X\neq Y$, then $G(n,X)\not\cong G(n,Y)$.

Without loss of generality, suppose that $C_{i_j}\in X\setminus Y$. Consider a copy of C_{i_j} in $G(n,Y)$. Circuits of different orders are pairwise non-embeddable; further, no circuit is an induced subgraph of a complete graph unless the circuit is C_3. However, each circuit in $G(n,Y)$ has at least four vertices by our initial assumptions (recall that $n\geq 2$ and the circuits have orders $\geq n+2$).

Hence, C_{i_j} is not an induced subgraph of $G(n,Y)\restriction G(n,Y)^0$. We can therefore, find a vertex of $G(n,Y)$ joined to every vertex in C_{i_j}. As this fails for any C_{i_j} in $G(n,X)^0$ by previous discussion, the graphs $G(n,X)$ and $G(n,Y)$ cannot be isomorphic.

Claim 2. The graph $G(n,X)$ is inexhaustible.

To see this, first name $G=G(n,X)$, and fix $x\in V(G)$. We show that $G-x\cong G$.

Case 1. The vertex x is in $V(G(n,X)^0)$.

In this case, we observe that by Theorem 3.2, $G(n,X)^0\cong G(n,X)^0-x$, say by an isomorphism f, since $G(n,X)^0$ is locally finite, age-closed, and with no infinite components. We prove that $G\cong G-x$ by extending f to an isomorphism via a back-and-forth argument (see Chapter 3 of [10]).

Suppose that we have already chosen a finite $A\leq G(n,X)$ not in $G(n,X)^0$ and a finite $B\leq G-x$ not in $G(n,X)^0-x$ so that $A\cup G(n,X)^0$ and $B\cup G(n,X)^0-x$ are isomorphic by an isomorphism f' extending f. Fix $y\in V(G)\setminus(V(G(n,X)^0)\cup V(A))$. By construction, y is joined to only finitely many vertices T of $G(n,X)^0$; y is joined to the vertices in A in some way. If we consider $D=f(T)\cup B$, then by construction, we should be able to find a z extending D in the same way that y extends $C=T\cup A$. The only possible obstruction would be if z must extend a circuit or clique in a way forbidden in the construction of $G(n,X)$. But then if we take preimages of this circuit or clique, then y would realize a forbidden extension in $G(n,X)$.

The argument for "going back" is similar, noting that since $x\in V(G(n,X)^0)$ the y chosen will never be x.

Case 2. The vertex x is not in $V(G(n,X)^0)$.

The argument is similar to Case 1; this time f is the identity map. The "going back" argument is similar to Case 1. For the "going forth" argument, we note that vertices extending a given subset in $G(n,X)^m$ are never unique (each n-e.c. condition can be witnessed by infinitely many distinct vertices). This finishes the proof of Claim 2. ∎

We close this section by showing that the class of inexhaustible graphs (of any order) is badly behaved from another vantage point; namely, we prove the following lemma. Recall that a class of graphs $\mathcal{K}$ is *first-order axiomatizable* if there is some language L and some set of L-sentences whose models are precisely the elements of $\mathcal{K}$.

Lemma 4.2. *The class $\mathcal{I}$ of inexhaustible graphs (of any order) has the following properties.*

1. *$\mathcal{I}$ is not first-order axiomatizable.*
2. *$\mathcal{I}$ is not closed under unions of chains.*

Proof. a) Our proof relies on some model-theoretic machinery. We show that $\mathcal{I}$ is not closed under elementary substructures (see p. 54 of [10]); this is sufficient to show $\mathcal{I}$ is not first-order axiomatizable. Let T^∞ denote the unique $\aleph_0$-regular tree. For all $x \in V(T^\infty)$, $T^\infty - x \cong \aleph_0 \cdot T^\infty$; in particular, T^∞ is not inexhaustible.

Suppose that $\mathcal{I}$ is closed under the taking of elementary substructures. Now, T^∞ is the unique existentially closed (e.c.) forest. (For background on e.c. structures the reader is directed to Chapter 8 of [10].) It is well-known that every infinite forest has a countable e.c. induced forest as an elementary substructure, and so must have T^∞ as an elementary substructure. The forest $\aleph_0 \cdot T^\infty$ is inexhaustible. Thus, if $\mathcal{I}$ were first-order axiomatizable, T^∞ would be inexhaustible, which is a contradiction.

b) We construct a union of a chain of graphs isomorphic to T^∞, so that each member of the chain is isomorphic to $G_0 = \aleph_0 \cdot T^\infty$. Name the components of G_0 as $T(i,j)$, with $i,j \geq 1$. Fix a vertex $y_{i,j}$ in $T(i,j)$. Let x be a vertex not in $V(G_0)$. For $n \geq 1$, define G_n to be the supergraph of G_0 formed by adding x to G_0 and joining x exactly to the vertices $y_{i,j}$, with $1 \leq i \leq n$, for all $j \geq 1$. Then $G_n \cong G_0$, since G_n is a forest with only infinite components, infinitely many components, and has each vertex of infinite degree. Let f_n be the identity embedding of G_n into G_{n+1}. The union of the chain $((G_n, f_n) : n \geq 0)$ is isomorphic to T^∞. ∎

5. Inexhaustible pseudo-homogeneous graphs

The definitive result on inexhaustible homogeneous graphs is the following theorem. Recall that a class $\mathcal{K}$ of finite graphs has the *strong amalgamation property*, or (SAP) if for all A, B, C in $\mathcal{K}$ and embeddings $f : A \to B$, $g : A \to C$, there is a $D\in\mathcal{K}$ and embeddings $h:B\to D$ and $i:C\to D$ so that $hf=ig$, and if $b\in V(B)$, $c\in V(C)$ and $h(b)=i(c)$, then there exists an $a\in V(A)$ such that $b=f(a)$ and $c=g(a)$. The graph D is called a *strong amalgam* of B and C over A. Informally, we can "glue" B and C together over A without identifying any vertices in $B-A$ with vertices in $C-A$.

Theorem 5.1 (El-Zahar, Sauer [7]). *A countable homogeneous graph G is inexhaustible if and only if $age(G)$ has (SAP).*

From this theorem and the theorem of Lachlan and Woodrow [12], the countable homogeneous inexhaustible graphs can be given: they are precisely the disjoint unions of infinite complete graphs, Henson's K_n-free graphs, for each integer $n\geq 2$, the complements of these, and the infinite random graph.

Fraïssé studied a weakening of homogeneity, which he referred to as pseudo-homogeneity (see Chapter 11 of [8]). We consider the case when $\mathcal{K}$ is closed under isomorphisms, induced subgraphs and disjoint unions. Let $\mathcal{C}$ be a subclass of $\mathcal{K}$ of finite graphs satisfying (SAP), the *joint embedding property* (for all $A,B\in\mathcal{C}$ there is a $C\in\mathcal{C}$ so that $A,B\leq C$), and satisfying *cofinality*: each finite graph in $\mathcal{K}$ can be embedded in a member of $\mathcal{C}$. Fraïssé proved that there is then a countable graph in $\mathcal{K}$, M, so that M is unique with the following properties; $\mathcal{K}$ in this case is called a *pseudo-amalgamation* class.

(PH1). The graph M embeds each graph in $\mathcal{C}$.
(PH2). Each finite $S\leq M$ is contained in $T\leq M$ with $T\in\mathcal{C}$.
(PH3). (amalgamating into) For each $G\leq M$ with $G\in\mathcal{C}$ and for each graph $H\in\mathcal{C}$ so that $G\leq H$, there is an $H'\leq M$ and an isomorphism $f:H\to H'$ such that $f\restriction G$ is the identity map on G.

M is referred to as *pseudo-homogeneous relative to $\mathcal{C}$ and $\mathcal{K}$*. Note that M is homogeneous if $\mathcal{C}=\mathcal{K}$.

Definition 5.2. 1. Let $A\leq B\in\mathcal{K}$. We say that $C\leq B$ *avoids* A if $V(C)\cap V(A)=\emptyset$.
2. The class $\mathcal{C}$ is *special* if for each finite $A\in\mathcal{C}$, and all $S<A$, there are $C',C\in\mathcal{C}$ so that $A\leq C$, $S\leq C'\leq C$ so that C' avoids $C\restriction(V(A)\setminus V(S))$. (See Figure 3 below.)

If $V(A)=\{x\}$ we will abuse notation and say that a subgraph avoids x. Roughly said, $\mathcal{C}$ is special if a version of (PH2) holds but with the freedom to avoid fixed subsets.

Theorem 5.3. *A countable pseudo-homogeneous graph G, relative to classes $\mathcal{C}\subseteq\mathcal{K}$, is inexhaustible if and only if $\mathcal{C}$ has (SAP) and is special.*

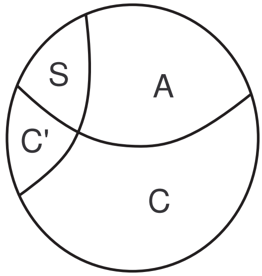

Figure 3. Item (2) of the definition of a special class.

If $\mathcal{K}$ is the class of forests, then $\mathcal{K}$ is a pseudo-amalgamation class with $\mathcal{C}$ the class of trees. However, $\mathcal{C}$ is not special. To see this, let A be the path with three vertices a,b,c, and let S be the end-vertices a,c of A. No extension of S to a tree can avoid b without inducing a circuit. Hence, by Theorem 5.3, the countable pseudo-homogeneous forest is not inexhaustible. We note that the countable pseudo-homogeneous forest is the $\aleph_0$-regular tree (see Section 11.6 of [8]), which we have already seen in Section 3 is not inexhaustible.

Proof of Theorem 5.3. ($\Rightarrow$) Let $A,B,C\in\mathcal{C}$ so that $A\leq B,C$ and $V(B)\cap V(C)=V(A)$. First embed B in G by (PH1), and consider C extending the copy of A in B. If we delete $S=V(B)\setminus V(A)$, then $G-S\cong G$, and hence, $G-S$ is pseudo-homogeneous. By (PH3), we can then amalgamate C into $G-S$ over A as C'. Then $V(C')\cap V(B)=V(A)$ in G. Now by (PH2) for G, extend $B\cup C'$ to an induced subgraph of $D\in\mathcal{C}$; D is a strong amalgam of B and C over A. (This is essentially the proof of one direction of Theorem 2 in [7].)

Now fix $A\in\mathcal{C}$, and fix $S<A$. Then A embeds in G by (PH1). If we delete $T=V(A)\setminus V(S)$, then $G-T\cong G$, and hence, $G-T$ is pseudo-homogeneous. By (PH2) we may extend S in $G-T$ to $C'\in\mathcal{C}$. Note that C' avoids $G\restriction T$ in G. By (PH2), choose C to be any finite induced subgraph of G in $\mathcal{C}$ that extends $C'\cup A$.

($\Leftarrow$) Fix $x\in V(G)$. We verify each of the axioms (PHn), $n=1,2,3$ for $G-x$, and thereby prove that $G-x\cong G$.

For (PH1), fix $A\in\mathcal{C}$, and embed A in G as A'. If $V(A')$ does not contain x, we are done. If $V(A')$ does contain x, then consider B defined to be the disjoint union of a copy A'' of A with G. Since $\mathcal{K}$ is closed under unions, B is in $\mathcal{K}$, so we may extend $B\restriction(V(A')\cup V(A''))$ to $B'\in\mathcal{C}$. By (PH3) for G, there is a copy of A'', say A''', in G disjoint from A. Then $V(A''')$ does not contain x.

For (PH2), consider $A\in\mathcal{K}$ embedded in $G-x$. Extend A to $B\in\mathcal{C}$ in G; if $V(B)$ does not contain x we are done. Otherwise, since $\mathcal{C}$ is special we may find $C',C\in\mathcal{C}$ so that $B\leq C$, $A\leq C'\leq C$ and $V(C')\cap V(B)=V(A)$. Using (PH3) we may embed C into G over B; the copy of C' in G avoids x.

For (PH3), consider $A,B\in\mathcal{C}$ so that $A\leq G-x$ and $A\leq B$. Extend $G\restriction(V(A)\cup\{x\})$ to $A'\in\mathcal{C}$ in G by (PH2). Form a strong amalgam D of A' and B over A so D is in $\mathcal{C}$. Then we may amalgamate D into G by (PH3). Then the copy of $B\leq D$ extending A in G avoids x. ∎

We note that the age of every homogeneous graph (where $\mathcal{K}=\mathcal{C}$) is special (choose $C'=S$ and $C=A$), so we recover a special case of Theorem 2 of [7]. Theorem 5.3 has the following interesting application. For a fixed finite graph G, we say that H is *G-colourable* if there is a homomorphism from H into G (an edge-preserving vertex-mapping). Note that a graph is n-colourable if and only if it admits a homomorphism into K_n. We may assume that G is a *core*: every endomorphism of G is onto. It is well-known (and easy to show) that the class of all G-colourable graphs, written $C(G)$, is closed under induced subgraphs and disjoint unions. The graph H is *uniquely G-colourable* if H is G-colourable, every homomorphism from H to G is onto, and for all homomorphisms f,h from H to G, there is an automorphism g of G so that $f=gh$. It was proven in [2] that $C(G)$ contains a countable pseudo-homogeneous graph, $M(G)$, with $\mathcal{C}$ in this case being the class of uniquely G-colourable graphs. (We note that $M(G)$ is homogeneous if and only if $G=K_1$.) Cofinality holds by the following construction. Let $H\in C(G)$ and let $f:H\to G$ be a fixed homomorphism. Define the *fixation* of H by f, $G(H,f)$, to have vertices $V(H)\cup V(G)$, and edges those of H and G, and additional edges xy, where $x\in V(H)$, and y is a neighbour of $f(x)$. Then $G(H,f)$ is uniquely G-colourable.

Corollary 5.4. *For all core graphs G, the graph $M(G)$ is inexhaustible.*

Proof. If $G=K_1$, then $M(G)=\overline{K_{\aleph_0}}$. Fix a non-trivial core G, and let $\mathcal{C}$ be the class of uniquely G-colourable graphs. By previous remarks, it is enough to show that $\mathcal{C}$ is special. Fix $A\in\mathcal{C}$, and fix $S<A$. For a fixed homomorphism f from A into G, let $C=G(A,f)$. Let $C'=C\restriction(V(G)\cup V(S))$. Then $C'=G(S,f\restriction S)$, and so C' is the desired uniquely G-colourable extension of S in C. ∎

6. Acknowledgements

The authors would like to thank the anonymous referees for their helpful suggestions.

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